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Boundedness of global positive solutions of a porous medium equation with a moving localized source

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Abstract

In this paper a porous medium equation with a moving localized source $u_t = u^r(\Delta u + af(u(x_0(t),t)))$ is considered. It is shown that under certain conditions solutions of the above equation blow up in finite time for large a or large initial data while there exist global positive solutions to the above equation for small a or small initial data. Moreover, in one space dimension case, it is also shown that all global positive solutions of the above equation are uniformly bounded, and this differs from that of a porous medium equation with a local source.
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1. Introduction

In this paper we investigate the boundedness of global positive solutions to the following initial boundary value problem:

$$\begin{aligned} &u_t = u^r\big(\Delta u + af\big(u\big(x_0(t),t\big)\big)\big), \quad x \in \Omega,\ t>0, \\ &u(x,t)=0, \quad x \in \partial\Omega,\ t>0, \qquad u(x,0)=u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

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where $0 < r < 1$ and $a > 0$ are constants, Ω is a bounded domain in $\mathbb{R}^n$ and is assumed to be with $C^{3+\alpha}$ boundary $\partial\Omega$ for some $\alpha \in (0, 1)$ throughout this paper, and $x_0(t)$ $(t \geqslant 0)$ is a moving point in Ω. By transformations $v = u^{1-r}$, $s = (1-r)t$, the equation in (1.1) becomes

$$v_s = \Delta v^m + af\big(v^m\big(x_0(ms), ms\big)\big), \quad x \in \Omega,\ s > 0, \tag{1.2}$$

here $m = 1/(1-r) > 1$. And it is well known that (1.2) is called the porous medium equation.

Problem (1.1) models a variety of physical phenomena, which arise, for example, in the study of the flow of a fluid through a porous medium with an internal moving localized source or in the study of population dynamics (see [3,6]).

Porous medium equations and the equations of porous medium type with or without local sources have been studied by a large number of authors since 70s in the last century (see [2,9,10, 14,15]).

Over the last two decades, much effort has been devoted to the study of the boundedness of global solutions of the following parabolic equation with local source term:

$$\begin{aligned} &u_t = u^r\big(\Delta u + u^p\big), \quad x \in \Omega,\ t > 0,\\ &u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{1.3}$$

see [7,16,18,22,24] and references therein. In order to summarize, just say here that if $n \leqslant 2$ or $p < (n+2)/(n-2)$ then all global solutions are uniformly bounded, whereas if $n \geqslant 3$ and $p \geqslant (n+2)/(n-2)$ then there exist unbounded global weak solutions, i.e.,

$$\sup_{t \geqslant 0} \big\| u(\cdot,t) \big\|_{L^\infty(\Omega)} = \infty.$$

Moreover, some unbounded global classical solutions even exist when $r = 0$, $p = (n+2)/(n-2)$ and Ω is a ball, see [11]. But for problem (1.1), we will show that all global positive solutions are uniformly bounded, that is to say, blow-up in infinite time can never occur, and this differs from that of a porous medium equation with a local source term.

Recently, for problem (1.1) with the equation replaced by the nondegenerate one

$$u_t = u_{xx} + u^p\big(x_0(t), t\big), \quad x \in (a,b),\ t > 0,$$

the boundedness of global solutions has been shown by Rouchon in [20], and it is different from that of the heat equation with a local source. Motivated by the results of the papers [7,20], we slightly modify the method developed by Rouchon and extend the results of [20] to the degenerate parabolic equation case. And for problem (1.1) with the equation replaced by the same localized equation with a fixed source

$$u_t = u^r\big(\Delta u + af\big(u(x_0,t)\big)\big), \quad x \in \Omega,\ t > 0,$$

the boundedness of global positive solutions has been shown in [4] under very strong conditions on the initial datum u_0, which are needed to ensure that the regularity of the solution u of the above equation is of class $C^{2+\alpha}$ up to $t = 0$ and that $u_t \geqslant 0$. But for the solution of problem (1.1), we cannot get these two properties which are really not needed here. And the method used in [4] to prove the uniform boundedness of global positive solutions cannot work here, we need another one—the method of using the concavity property of the specified function constructed later in Section 4.

Before stating our main results, we make some assumptions on $x_0(t)$, $u_0(x)$ and $f(s)$ as follows:

(H$_1$) $x_0 : \mathbb{R}^+ \to \Omega$ is continuously differentiable, and $x_0(t) \in K$ for all $t \geqslant 0$, where K is a fixed compact subset of Ω.
(H$_2$) $u_0 \in C^1(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ for some $\alpha \in (0,1)$, $u_0(x) > 0$ in Ω, $u_0(x) = 0$ on $\partial\Omega$ and $\frac{\partial u_0}{\partial \nu}|_{\partial\Omega} < 0$, where ν is the unit outward normal vector on $\partial\Omega$.
(H$_3$) $f \in C([0,\infty)) \cap C^1(0,\infty)$ such that $f(0) \geqslant 0$ and $f'(s) > 0$ in $(0,\infty)$.
(H$_4$) f is convex in $(0,\infty)$ and $\int_{s_0}^{\infty} \frac{ds}{f(s)} < \infty$ for some $s_0 \geqslant 0$.

Now, let us state our main results.

Theorem 1.1. *Suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H$_1$)–(H$_4$), *then there exists a global positive solution to problem* (1.1) *if a is small enough or $f(s) = o(s)$ as $s \to 0$ and $u_0(x)$ is sufficiently small. While if a or $u_0(x)$ is sufficiently large then the solution of problem* (1.1) *blows up in finite time.*

Theorem 1.2. *Suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H$_1$)–(H$_4$). *And assume that $u(x,t)$ is a global positive solution of problem* (1.1) *in one space dimension, then $u(x,t)$ is uniformly bounded in space and time, that is,* $\sup_{t\geqslant 0} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$.

Remark 1.3.

(i) Since for all $t \geqslant 0$, $x_0(t) \in K$ and $K \subset \Omega$ is a fixed compact subset, we can call the source $af(u(x_0(t),t))$ the moving localized source. And also due to the compactness of the subset K of Ω, we can show the blow-up result by using the upper and lower solutions' technique and can get the uniform boundedness of global positive solutions of problem (1.1) by using the concavity property of the constructed function.
(ii) The solutions we considered here are classical, but one can equally well consider weak solutions and use the Stampcchia's treatment of u_+ (see for example [23]) and the comparisons under possibly weaker regularity hypotheses (see for example [1,5]), etc.

This paper is organized as follows. In Section 2, we establish the local existence of the classical positive solution of problem (1.1). Results regarding to global existence and finite time blow-up for problem (1.1) are presented in Section 3. In Section 4, we show the uniform boundedness of global positive solutions.

2. Local existence and comparison principle

We set $Q_T = \Omega \times (0,T]$, $Q_{\eta,T} = \Omega \times (\eta,T]$, and $S_T = \partial\Omega \times (0,T]$. First, we need the following comparison principle.

Lemma 2.1. *Assume that $w(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ and satisfies*

$$\begin{aligned}
&w_t - d(x,t)\Delta w \geqslant \sum_{i=1}^{n} b_i(x,t) w_{x_i} + c_1(x,t) w + c_2(x,t) w\big(x_0(t),t\big), \quad (x,t) \in Q_T,\\
&w(x,t) \geqslant 0, \quad (x,t) \in S_T,\\
&w(x,0) \geqslant 0, \quad x \in \Omega, \qquad (2.1)
\end{aligned}$$

where $d(x,t)$ and $b_i(x,t)$ $(i = 1, \dots, n)$ are continuous in Q_T, $c_1(x,t)$ and $c_2(x,t)$ are bounded and continuous in Q_T, $c_2(x,t)$, $d(x,t) \geqslant 0$ in Q_T, and $x_0 : \mathbb{R}^+ \to \Omega$ is continuous. Then $w(x,t) \geqslant 0$ on $\overline{Q}_T$.

Proof. The proof is similar to the classical case, see [19, Lemma 2.2.1]. We omit it here. □

In order to get the global existence and finite time blow-up results for problem (1.1), we need the following comparison principle, and it is a direct consequence of Lemma 2.1.

Lemma 2.2. *Assume that $\tilde{u} \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ is an upper solution of problem* (1.1) *and $\hat{u} \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ is a nonnegative lower solution of problem* (1.1)*, and that there exists a small positive constant η such that $\tilde{u}(x,t) \geqslant \eta$ on $\overline{Q}_T$. Then $\tilde{u}(x,t) \geqslant \hat{u}(x,t)$ on $\overline{Q}_T$.*

To show the local solvability of problem (1.1), different from [4], we use the method of boundary regularization and consider the following regularized problem:

$$\begin{aligned}
&u_{\varepsilon t} = u_\varepsilon^r\big(\Delta u_\varepsilon + af\big(u_\varepsilon\big(x_0(t),t\big)\big)\big), \quad x \in \Omega,\ t > 0,\\
&u_\varepsilon(x,t) = \varepsilon, \quad x \in \partial\Omega,\ t > 0,\\
&u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad x \in \Omega,
\end{aligned} \tag{2.2}$$

where $0 < \varepsilon \leqslant 1$. Noticing that $\partial\Omega$ is assumed to be of class $C^{3+\alpha}$ at the beginning of Section 1 and using the classical theories of parabolic equations (see [8,13,17]) and the comparison principle given by Lemma 2.1, although the first order of compatibility condition has not been satisfied, we can show in much the same way as in [21, Theorem A.4] the following result.

Theorem 2.3. *Assume that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3). *Then there exists a maximal in time function $u_\varepsilon(x,t) \geqslant \varepsilon$, defined on $\overline{\Omega} \times [0, T_\varepsilon^*)$ for some $T_\varepsilon^* \in (0,\infty]$, such that for all $0 < \eta < T < T_\varepsilon^*$, $u_\varepsilon \in C^1(\overline{Q}_T) \cap C^{2+\alpha}(\overline{Q}_{\eta,T})$, and u_ε is the unique classical solution of problem* (2.2). *Moreover, if $T_\varepsilon^* < \infty$ then $\limsup_{t \to T_\varepsilon^*} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.*

Using the comparison principle given by Lemma 2.1, we can show that $u_\varepsilon(x,t)$ has the following monotonicity with respect to ε.

Lemma 2.4. *Let $1 \geqslant \varepsilon_1 > \varepsilon_2 > 0$. And suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3)*, and that u_{ε_1} and u_{ε_2} are solutions of problem* (2.2) *with ε_1 and ε_2, respectively. Then $u_{\varepsilon_1}(x,t) \geqslant u_{\varepsilon_2}(x,t)$ on $\overline{\Omega} \times [0, T_{\varepsilon_1}^*)$ and $T_{\varepsilon_1}^* \leqslant T_{\varepsilon_2}^*$.*

Proof. Let $w(x,t) = u_{\varepsilon_1}(x,t) - u_{\varepsilon_2}(x,t)$, then w satisfies

$$\begin{aligned}
&w_t - u_{\varepsilon_2}^r \Delta w = r\eta^{r-1}\big(\Delta u_{\varepsilon_1} + af\big(u_{\varepsilon_1}\big(x_0(t),t\big)\big)\big)w + au_{\varepsilon_2}^r f'\big(\xi\big(x_0(t),t\big)\big)w\big(x_0(t),t\big),\\
&\quad (x,t) \in \Omega \times \big(0, T_{\varepsilon_1}^*\big),\\
&w(x,t) = \varepsilon_1 - \varepsilon_2 > 0, \quad (x,t) \in \partial\Omega \times \big(0, T_{\varepsilon_1}^*\big),\\
&w(x,0) = \varepsilon_1 - \varepsilon_2 > 0, \quad x \in \Omega,
\end{aligned} \tag{2.3}$$

where $\eta \geqslant \varepsilon_2$ and $f'(\xi(x_0(t),t)) \geqslant 0$. Therefore Lemma 2.1 implies that $w(x,t) \geqslant 0$, that is, $u_{\varepsilon_1}(x,t) \geqslant u_{\varepsilon_2}(x,t)$ on $\overline{\Omega} \times [0, T_{\varepsilon_1}^*)$, and therefore we have $T_{\varepsilon_1}^* \leqslant T_{\varepsilon_2}^*$. □

It follows from Lemma 2.4 that u_ε are monotone with respect to ε and

$$0 \leqslant u_\varepsilon(x,t) \leqslant u_1(x,t), \quad (x,t) \in \overline{\Omega} \times \big[0, T_1^*\big). \tag{2.4}$$

And therefore the pointwise limit

$$u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t), \quad (x,t) \in \overline{\Omega} \times \big[0, T^*\big), \tag{2.5}$$

exists, where $T^* = \lim_{\varepsilon \to 0} T_\varepsilon^*$.

Let $\lambda_1 > 0$ and $\varphi_1(x)$ be the first eigenvalue and the corresponding eigenfunction of the following elliptic problem:

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{2.6}$$

where $\varphi_1(x)$ is chosen such that $\varphi_1(x) > 0$ in Ω and $\max_{x \in \overline{\Omega}} \varphi_1(x) = 1$. Since $\partial\Omega$ is assumed to be of class $C^{3+\alpha}$, it belongs to C^1. And then it follows from (H_2) that there exists a small enough positive constant k such that

$$k\varphi_1(x) \leqslant u_0(x). \tag{2.7}$$

To prove that $u(x,t)$ defined by (2.5) is a positive classical solution of problem (1.1), we need yet the following lemma which gives a uniform lower bound for u_ε.

Lemma 2.5. *Suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3). *Let $\phi(x,t) = k\varphi_1(x)e^{-\rho t}$, where $k > 0$ is given by* (2.7), *and $\rho = \lambda_1 k^r$. Then for all $\varepsilon \in (0,1]$, the solution $u_\varepsilon(x,t)$ of problem* (2.2) *satisfies $u_\varepsilon(x,t) \geqslant \phi(x,t)$ on $\overline{\Omega} \times [0, T_\varepsilon^*)$.*

Proof. Substituting $\phi(x,t)$ into Eq. (2.2), we obtain

$$\begin{aligned}
&\phi_t - \phi^r \Delta\phi - a\phi^r f\big(\phi\big(x_0(t), t\big)\big) \\
&\quad = -k\rho\varphi_1(x)e^{-\rho t} - k^r\varphi_1^r(x)e^{-r\rho t}\big(-k\lambda_1\varphi_1(x)e^{-\rho t}\big) \\
&\qquad - ak^r\varphi_1^r(x)e^{-r\rho t} f\big(k\varphi_1\big(x_0(t)\big)e^{-\rho t}\big) \\
&\quad \leqslant k\varphi_1(x)e^{-\rho t}\big(\lambda_1 k^r \varphi_1^r(x)e^{-r\rho t} - \rho\big) \\
&\quad \leqslant k\varphi_1(x)e^{-\rho t}\big(\lambda_1 k^r - \rho\big) = 0, \quad (x,t) \in \Omega \times \big(0, T_\varepsilon^*\big).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\phi(x,t) = 0 < \varepsilon, \quad (x,t) \in \partial\Omega \times \big(0, T_\varepsilon^*\big), \\
&\phi(x,0) = k\varphi_1(x) < u_0(x) + \varepsilon, \quad x \in \overline{\Omega}.
\end{aligned}$$

All the above inequalities show that $\phi(x,t)$ is a lower solution of problem (2.2), then Lemma 2.2 implies that $\phi(x,t) \leqslant u_\varepsilon(x,t)$, $(x,t) \in \overline{\Omega} \times [0, T_\varepsilon^*)$. □

Theorem 2.6. *Suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3). *Then the function $u(x,t)$ defined by* (2.5) *is the unique classical positive solution of problem* (1.1) *in $\Omega \times (0, T^*)$ with $u(x,t) \geqslant \phi(x,t)$. Moreover, if $T^* < +\infty$, then* $\limsup_{t \to T^*} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

Proof. The argument is standard, and therefore it is omitted here. □

3. Global existence and finite time blow-up

In this section we show the global existence and finite time blow-up results for problem (1.1). And Theorem 1.1 can be proved in much the same way as in [4], however the result here is proved for the case of a porous medium equation with a moving localized source. Therefore we give the proof for the ease of readers and for the sake of completeness. We divide the proof of Theorem 1.1 into the following two theorems.

Theorem 3.1. *Suppose that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3). *Then there exists a global positive solution $u(x,t)$ to problem* (1.1) *if a is small enough or if $f(s)=o(s)$ as $s\to 0$ and $u_0(x)$ is sufficiently small.*

Proof. Let $\psi(x)$ be the unique positive solution of the following elliptic problem:

$$-\Delta\psi(x)=1,\quad x\in\Omega;\qquad \psi(x)=L_0,\quad x\in\partial\Omega, \tag{3.1}$$

where L_0 is a small positive constant. Then $\psi\in C(\overline{\Omega})\cap C^2(\Omega)$. By the maximum principle of elliptic equations, we know that $\psi(x)\geqslant L_0$, $x\in\overline{\Omega}$. Set $M=\max_{x\in\overline{\Omega}}\psi(x)$, then we have $M>L_0$. Define $\tilde u(x,t)=L\psi(x)$, where L is a positive constant to be fixed later. Since from the assumption (H_1) we know that $\{x_0(t)\mid t\geqslant 0\}\subset K\Subset\Omega$, a series of computations yields

$$\begin{aligned}\tilde u_t-\tilde u^r\big(\Delta\tilde u+af\big(\tilde u\big(x_0(t),t\big)\big)\big)&=-L^r\psi^r(x)\big(-L+af\big(L\psi\big(x_0(t)\big)\big)\big)\\&\geqslant L^r\psi^r(x)\big(L-af(LM)\big).\end{aligned}\tag{3.2}$$

(i) Choose $L>0$ such that $LL_0\geqslant\max_{x\in\overline{\Omega}}u_0(x)$ and set $a_0=L/f(LM)$. Then for $a\leqslant a_0$, from (3.2) we get

$$\begin{aligned}&\tilde u_t-\tilde u^r\big(\Delta\tilde u+af\big(\tilde u\big(x_0(t),t\big)\big)\big)\geqslant L^r\psi^r(x)\big(L-af(LM)\big)\geqslant 0,\quad x\in\Omega,\ t>0,\\&\tilde u(x,t)=LL_0>0,\quad x\in\partial\Omega,\ t>0,\\&\tilde u(x,0)=L\psi(x)\geqslant LL_0\geqslant u_0(x),\quad x\in\Omega.\end{aligned}\tag{3.3}$$

And (3.3) shows that $\tilde u$ is an upper solution of problem (1.1). Since $\tilde u(x,t)\geqslant LL_0>0$ on $\overline{\Omega}\times[0,+\infty)$, Lemma 2.2 and Theorem 2.6 imply that there exists a global positive solution $u(x,t)$ to problem (1.1) if $a\leqslant a_0$.

(ii) If $f(s)=o(s)$ as $s\to 0$, then there exists a small positive constant δ_0 such that for all $s\in(0,\delta_0]$,

$$\frac{f(s)}{s}\leqslant\frac{1}{aM}.\tag{3.4}$$

Let $L_0>0$ such that $L_0<(\sqrt{M_0^2+4\delta_0^2}-M_0)/2$, where M_0 is the maximum of $\psi_0(x)$ on $\overline{\Omega}$ and $\psi_0(x)$ is the unique solution of the elliptic problem (3.1) with the boundary condition replaced by the homogeneous one, then we have $L_0<\delta_0/M_0$. And choose $L=\delta_0/M$, then from (3.4), we have

$$\frac{f(LM)}{LM}\leqslant\frac{1}{aM}.\tag{3.5}$$

If $u_0(x)$ is small enough such that $0\leqslant u_0(x)\leqslant LL_0$, then by combining (3.2) and (3.5), we can easily verify that $\tilde u(x,t)$ also satisfies the inequalities in (3.3) and therefore it is an upper solution of problem (1.1). Also due to $\tilde{u}(x,t) \geqslant LL_0 > 0$, Lemma 2.2 and Theorem 2.6 guarantee that there exists a positive global solution $u(x,t)$ to problem (1.1) if $\lim_{s\to 0} \frac{f(s)}{s} = 0$ and $u_0(x)$ is sufficiently small. □

Theorem 3.2. *Assume that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H$_1$)–(H$_4$), *then the positive solution $u(x,t)$ of problem* (1.1) *blows up in finite time provided a is sufficiently large or $u_0(x)$ is large enough.*

Proof. Since $x_0(t) \in K$ for all $t \geqslant 0$ and K is a fixed compact subset of Ω, there exists a domain Ω_1 such that $K \subset \Omega_1 \Subset \Omega$. Let μ_1 be the first eigenvalue of the following elliptic problem:

$$-\Delta\phi(x) = \mu\phi(x), \quad x \in \Omega_1; \qquad \phi(x) = 0, \quad x \in \partial\Omega_1, \tag{3.6}$$

and let $\phi_1(x)$ be the corresponding eigenfunction with $\phi_1(x) > 0$ in Ω_1 and $\max_{x\in\Omega_1} \phi_1(x) = 1$. Set $\hat{u}(x,t) = \lambda z^{\frac{1}{1-r}}(t)\phi_1(x)$, where $\lambda > 0$ is a constant and $z(t)$ is a positive nondecreasing function to be fixed later. Substituting $\hat{u}(x,t)$ into (1.1), and a series of computations yields

$$\begin{aligned}
&\hat{u}_t - \hat{u}^r\big(\Delta\hat{u} + af\big(\hat{u}\big(x_0(t),t\big)\big)\big) \\
&\quad = \frac{1}{1-r}\lambda z^{\frac{r}{1-r}}(t)z'(t)\phi_1(x) \\
&\qquad - \lambda^r z^{\frac{r}{1-r}}(t)\phi_1^r(x)\big(-\lambda\mu_1 z^{\frac{1}{1-r}}(t)\phi_1(x) + af\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big)\big) \\
&\quad \leqslant \lambda^r z^{\frac{r}{1-r}}(t)\phi_1^r(x)\left(\frac{1}{1-r}\lambda^{1-r}z'(t) - \frac{a}{2}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big)\right) \\
&\qquad + \lambda^r z^{\frac{r}{1-r}}(t)\phi_1^r(x)\left(\lambda\mu_1 z^{\frac{1}{1-r}}(t) - \frac{a}{2}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big)\right).
\end{aligned} \tag{3.7}$$

(i) From Lemma 2.5 and (2.5), we know that $u(x,t) \geqslant k\varphi_1(x)e^{-\rho t}$ on $\overline{\Omega} \times [0,T^*)$, where k and ρ are positive constants given by Lemma 2.5 and $\varphi_1(x)$ is the first eigenfunction of the eigenvalue problem (2.6) with $\varphi_1(x) > 0$ in Ω and $\max_{x\in\Omega}\varphi_1(x) = 1$. Let $m_0 = \min_{x\in\overline{\Omega}_1}\varphi_1(x)$, then $m_0 > 0$. Set $\delta = km_0e^{-\rho T_z^*}$, then $\delta > 0$ and $u(x,t) \geqslant \delta$ on $\overline{\Omega}_1 \times [0,\min\{T^*, T_z^*\})$, where T_z^* is given by the following equations (3.9) and (3.10). Choose $\lambda \in (0,\delta)$, and let $z(t)$ be the solution of the following initial value problem:

$$\begin{aligned}
&z'(t) = \frac{1-r}{2}\lambda^{r-1}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big), \quad t > 0, \\
&z(0) = 1.
\end{aligned} \tag{3.8}$$

It follows from the condition $f(s) \geqslant 0$ in (H$_3$) that $z(t)$ is nondecreasing in t. Then $z(t) \geqslant 1$ for all $t \geqslant 0$. Since $x_0(t) \in K$ for all $t \geqslant 0$ and K is a compact subset of Ω_1, it follows that $m = \min_{x\in K}\phi_1(x) > 0$. Then $\phi_1(x_0(t)) \geqslant m$ for all $t \geqslant 0$. Using the assumption (H$_4$), we know that $z(t)$ is well defined on $[0,T_z^*)$, where

$$\begin{aligned}
T_z^* &= \frac{2}{1-r}\lambda^{1-r}\int_0^{T_z^*}\frac{z'(t)\,dt}{f(\lambda\phi_1(x_0(t))z^{\frac{1}{1-r}}(t))} \\
&\leqslant \frac{2}{1-r}\lambda^{1-r}\int_0^{T_z^*}\frac{z'(t)\,dt}{f(\lambda mz(t))}
\end{aligned}$$

$$= \frac{2}{1-r}\lambda^{-r}m^{-1}\int_{\lambda m}^{\lambda m z(T_z^*)}\frac{ds}{f(s)}$$

$$\leqslant \frac{2}{1-r}\frac{1}{\lambda^r m}\int_{\lambda m}^{\infty}\frac{ds}{f(s)} < \infty \tag{3.9}$$

such that

$$\lim_{t\to T_z^*} z(t) = \infty. \tag{3.10}$$

Since $f(s)$ satisfies conditions (H_3) and (H_4), we claim that $\frac{f(s)}{s} \to \infty$ as $s \to \infty$. In fact, from the condition $\int_{s_0}^{\infty}\frac{ds}{f(s)} < \infty$ we know that $\lim_{s\to\infty} f(s) = \infty$. Since $f(s)$ is convex in $(0,\infty)$, we know that $f'(s)$ is nondecreasing in $(0,\infty)$. Utilizing L'Hospital rule, we get

$$\lim_{s\to\infty}\frac{f(s)}{s} = \lim_{s\to\infty} f'(s).$$

We argue by contradiction that $\lim_{s\to\infty} f'(s) = N < \infty$. Then there exists $s_1 > s_0$ such that $f(s) \leqslant \frac{3}{2}Ns$ for all $s \geqslant s_1$, where s_0 is a nonnegative constant given in the assumption (H_4). And then

$$\int_{s_0}^{\infty}\frac{ds}{f(s)} \geqslant \frac{2}{3N}\int_{s_1}^{\infty}\frac{ds}{s} = \infty.$$

It is absurd. Hence $\lim_{s\to\infty}\frac{f(s)}{s} = \infty$. So there exists a constant $s_2 > s_0$ such that

$$\frac{f(s)}{s} > \frac{2\mu_1}{m} \tag{3.11}$$

for all $s \geqslant s_2$. For such s_2, there exists a $t_0 \in (0, T_z^*)$ such that

$$\lambda m z^{\frac{1}{1-r}}(t) \geqslant s_2 \tag{3.12}$$

for all $t \geqslant t_0$. Set

$$a_0 = \max\left\{1, \max_{t\in[0,t_0]}\frac{2\lambda\mu_1 z^{\frac{1}{1-r}}(t)}{f(\lambda m z^{\frac{1}{1-r}}(t))}\right\}, \tag{3.13}$$

then from (3.11)–(3.13), we get for $a \geqslant a_0$,

$$\lambda\mu_1 z^{\frac{1}{1-r}}(t) - \frac{a}{2}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big) \leqslant \lambda\mu_1 z^{\frac{1}{1-r}}(t) - \frac{a}{2}f\big(\lambda m z^{\frac{1}{1-r}}(t)\big) \leqslant 0. \tag{3.14}$$

Hence from (3.7), (3.8) and (3.14), we obtain for all $a \geqslant a_0$,

$$\begin{aligned}
&\hat{u}_t - \hat{u}^r\big(\Delta\hat{u} + af\big(\hat{u}\big(x_0(t),t\big)\big)\big) \\
&\quad \leqslant \lambda^r z^{\frac{r}{1-r}}(t)\phi_1^r(x)\left(\frac{1}{1-r}\lambda^{1-r}z'(t) - \frac{a}{2}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big)\right) \\
&\qquad + \lambda^r z^{\frac{r}{1-r}}(t)\phi_1^r(x)\left(\lambda\mu_1 z^{\frac{1}{1-r}}(t) - \frac{a}{2}f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big)\right) \\
&\quad \leqslant 0, \quad (x,t) \in \Omega_1 \times \big(0, \min\{T^*, T_z^*\}\big).
\end{aligned} \tag{3.15}$$

On the other hand, we also have

$$\hat{u}(x,t) = 0 < u(x,t), \quad x \in \partial\Omega_1 \times \big(0, \min\big\{T^*, T_z^*\big\}\big), \tag{3.16}$$

and

$$\hat{u}(x,0) = \lambda\phi_1(x) < \delta \leqslant u_0(x), \quad x \in \Omega_1. \tag{3.17}$$

Inequalities (3.15)–(3.17) show that for $a \geqslant a_0$, $\hat{u}(x,t)$ is a lower solution of problem (1.1) with the definition area $\Omega \times (0, T^*)$ replaced by $\Omega_1 \times (0, \min\{T^*, T_z^*\})$. Since $u(x,t) \geqslant \delta$ on $\overline{\Omega}_1 \times [0, \min\{T^*, T_z^*\})$, Lemma 2.2 implies that $u(x,t) \geqslant \hat{u}(x,t)$ on $\overline{\Omega}_1 \times [0, \min\{T^*, T_z^*\})$, hence $T^* \leqslant T_z^*$, and therefore $u(x,t)$ blows up in finite time T^* for large a.

(ii) From the above discussion we know that $\lim_{s\to\infty} \frac{f(s)}{s} = \infty$, hence there exists a positive constant $s_3 > s_0$ such that for $s \geqslant s_3$,

$$\frac{f(s)}{s} \geqslant \frac{2\mu_1}{am}. \tag{3.18}$$

Choose $\lambda > 0$ large enough such that $\lambda \geqslant \frac{s_3}{m}$, then $\lambda\phi_1(x_0(t)) \geqslant \lambda m \geqslant s_3$ for all $t \geqslant 0$. Define $z(t)$ as the solution of the following initial value problem:

$$\begin{aligned} &z'(t) = \frac{a}{2}(1-r)\lambda^{r-1} f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big), \quad t > 0, \\ &z(0) = 1. \end{aligned} \tag{3.19}$$

Since $f(s)$ satisfies (H_3), we know that $z(t)$ is nondecreasing in t. Then $z(t) \geqslant 1$ for $t \geqslant 0$. Using assumption (H_4), we know that $z(t)$ is well defined on a finite interval $[0, T_z^*)$, where

$$\begin{aligned} T_z^* &= \frac{2}{(1-r)a}\lambda^{1-r} \int_0^{T_z^*} \frac{z'(t)\,dt}{f(\lambda\phi_1(x_0(t))z^{\frac{1}{1-r}}(t))} \\ &\leqslant \frac{2}{(1-r)a}\lambda^{1-r} \int_0^{T_z^*} \frac{z'(t)\,dt}{f(\lambda m z(t))} \\ &= \frac{2}{(1-r)a}\lambda^{-r}(m)^{-1} \int_{\lambda m}^{\lambda m z(T_z^*)} \frac{ds}{f(s)} \\ &\leqslant \frac{2}{(1-r)a}\frac{1}{\lambda^r m} \int_{\lambda m}^{\infty} \frac{ds}{f(s)} < \infty \end{aligned} \tag{3.20}$$

such that $\lim_{t\to T_z^*} z(t) = \infty$. Then from the property of $z(t)$ and the choice of λ we get

$$\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t) \geqslant \lambda\phi_1\big(x_0(t)\big) \geqslant s_3, \quad t \in \big[0, T_z^*\big).$$

Hence from (3.18), we get

$$\lambda\mu_1 z^{\frac{1}{1-r}}(t) \leqslant \frac{a}{2} f\big(\lambda\phi_1\big(x_0(t)\big)z^{\frac{1}{1-r}}(t)\big), \quad t \in \big[0, T_z^*\big). \tag{3.21}$$

Hence from (3.7), (3.19) and (3.21), we obtain

$$\hat{u}_t - \hat{u}^r\big(\Delta\hat{u} + af\big(\hat{u}\big(x_0(t), t\big)\big)\big) \leqslant 0, \quad (x,t) \in \Omega_1 \times \big(0, \min\big\{T^*, T_z^*\big\}\big). \tag{3.22}$$

On the other hand, we also have

$$\hat{u}(x,t) = 0 < u(x,t), \quad (x,t) \in \partial\Omega_1 \times \big(0, \min\big\{T^*, T_z^*\big\}\big).$$

If we choose $u_0(x)$ large enough such that $u_0(x) \geqslant \lambda\phi_1(x)$ on $\overline{\Omega}_1$, then from the above inequality and (3.22) we know that $\hat{u}(x,t)$ is a lower solution of problem (1.1) with the definition area $\Omega \times (0, T^*)$ replaced by $\Omega_1 \times (0, \min\{T^*, T_z^*\})$. Since $u(x,t) \geqslant \delta$ on $\overline{\Omega}_1 \times [0, \min\{T^*, T_z^*\})$, Lemma 2.2 implies that $u(x,t) \geqslant \hat{u}(x,t)$ on $\overline{\Omega}_1 \times [0, \min\{T^*, T_z^*\})$, hence $T^* \leqslant T_z^*$, and therefore $u(x,t)$ blows up in finite time if the initial datum $u_0(x)$ is large enough. □

From Theorems 3.1 and 3.2, we completed the proof of Theorem 1.1.

4. Boundedness of global positive solutions in one space dimension

In this section we give out the proof of Theorem 1.2. Throughout this section, we assume that $u(x,t)$ is a global positive solution of problem (1.1) and that we are in one-dimensional space $\mathbb{R}^1$, that is to say, the maximal existence time $T^* = \infty$ and $\Omega \subset \mathbb{R}^1$. In order to simplify the proof, let us assume $\Omega = (a_1, b)$ and set $g(t) = af(u(x_0(t), t))$. We argue by contradiction, and suppose that

$$\limsup_{t\to\infty} \big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} = \infty. \tag{4.1}$$

Lemma 4.1. *Assume that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3)*, and that $u(x,t)$ is a global positive solution of problem* (1.1)*. Then under hypothesis* (4.1)*, we have*

$$\forall A > 0,\ \forall T > 0,\ \exists t_1 \geqslant T \quad \text{such that} \quad g(t_1) > A. \tag{4.2}$$

Proof. Assume that the conclusion of this lemma is not true, then there exist two positive constants A_0 and T_0 such that $g(t) \leqslant A_0$ for all $t \geqslant T_0$. Then it follows from the condition $u(x,t) > 0$ in $\Omega \times (0, \infty)$ that

$$u_t - u^r\Delta u \leqslant u^r A_0, \quad x \in \Omega,\ t > T_0. \tag{4.3}$$

Consider the following stationary problem:

$$\begin{aligned} &-\Delta w(x) = A_0, \quad x \in \Omega, \\ &w(x) = 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.4}$$

We know that the above linear elliptic problem admits a unique solution $w \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $w(x) > 0$ in Ω and $w(x) = 0$ on $\partial\Omega$ (see [12, Chapters 3 and 6]). Letting $v(x,t) = w(x) + \|u(x,T_0)\|_{L^\infty(\Omega)} + 1$ and using (4.4), we get

$$\begin{aligned} &v_t - v^r\Delta v = -v^r\Delta w = A_0 v^r, \quad x \in \Omega,\ t > T_0, \\ &v(x,t) \geqslant 1, \quad x \in \partial\Omega,\ t > T_0, \\ &v(x,T_0) = w(x) + \big\|u(x,T_0)\big\|_{L^\infty(\Omega)} + 1 \geqslant u(x,T_0), \quad x \in \Omega. \end{aligned}$$

Since $v(x,t) \geqslant 1$ for $x \in \overline{\Omega}$, $t \geqslant T_0$, it follows from the comparison principle which can be seen as a direct consequence of Lemma 2.1 that $u(x,t) \leqslant v(x,t)$ on $\overline{\Omega} \times [T_0, \infty)$. Hence $u(x,t) \leqslant$

$\sup_{t\geqslant T_0}\|v(\cdot,t)\|_{L^\infty(\Omega)} = C < \infty$ for $x \in \Omega$, $t > T_0$, and this contradicts to hypothesis (4.1). Therefore the conclusion (4.2) holds. □

Lemma 4.2. *Assume that $x_0(t)$, $u_0(x)$ and $f(s)$ satisfy the assumptions* (H_1)–(H_3)*, and that $u(x,t)$ is a global positive solution of problem* (1.1)*. Then for any given small positive constant $\eta_0 < T_1^*/2$, $u_{xx} \leqslant C_0$ on any compact subset Ω_2 of Ω for all $t \geqslant \eta_0$, where $C_0 = C_0(\eta_0)$ is a positive constant depending only on η_0.*

Proof. Let $w = u_{\varepsilon xx}$, and differentiate Eq. (2.2) with respect to x twice, we get

$$\begin{aligned} w_t &= u_\varepsilon^r w_{xx} + 2ru_\varepsilon^{r-1}u_{\varepsilon x}w_x + ru_\varepsilon^{r-1}\big(u_{\varepsilon xx} + af\big(u_\varepsilon\big(x_0(t),t\big)\big)\big)w \\ &\quad + r(r-1)u_\varepsilon^{r-2}\big(u_{\varepsilon xx} + af\big(u_\varepsilon\big(x_0(t),t\big)\big)\big)u_{\varepsilon x}^2 \\ &= u_\varepsilon^r w_{xx} + 2ru_\varepsilon^{r-1}u_{\varepsilon x}w_x + ru_\varepsilon^{-1}u_{\varepsilon t}w + r(r-1)u_\varepsilon^{r-2}u_{\varepsilon x}^2 w \\ &\quad + r(r-1)au_\varepsilon^{r-2}u_{\varepsilon x}^2 f\big(u_\varepsilon\big(x_0(t),t\big)\big), \quad (x,t) \in \Omega \times \big(\eta_0, T_\varepsilon^*\big). \end{aligned} \tag{4.5}$$

In view of $u_\varepsilon \geqslant \varepsilon$, $f(u_\varepsilon(x_0(t),t)) \geqslant f(\varepsilon) > 0$ and $r < 1$, we have

$$\begin{aligned} &w_t - u_\varepsilon^r w_{xx} - 2ru_\varepsilon^{r-1}u_{\varepsilon x}w_x - \big(ru_\varepsilon^{-1}u_{\varepsilon t} + r(r-1)u_\varepsilon^{r-2}u_{\varepsilon x}^2\big)w \leqslant 0, \\ &\quad (x,t) \in \Omega \times \big(\eta_0, T_\varepsilon^*\big). \end{aligned} \tag{4.6}$$

On the other hand, by using the fact that $u_{\varepsilon t} = 0$ on $\partial\Omega \times (\eta_0, T_\varepsilon^*)$, we also have

$$\begin{aligned} &w(x,t) = -af\big(u_\varepsilon\big(x_0(t),t\big)\big) < 0 \quad \text{on } \partial\Omega \times \big(\eta_0, T_\varepsilon^*\big), \\ &w(x,t) = u_{\varepsilon xx}(x,\eta_0) \quad \text{in } \Omega. \end{aligned} \tag{4.7}$$

From (4.6), (4.7), by using the maximum principle, we get

$$u_{\varepsilon xx}(x,t) = w(x,t) \leqslant \big\|u_{\varepsilon xx}(x,\eta_0)\big\|, \quad (x,t) \in \Omega \times \big[\eta_0, T_\varepsilon^*\big). \tag{4.8}$$

From Theorem 2.6 we know that $u_{\varepsilon xx}$ converges to u_{xx} uniformly on $\overline{\Omega}_2 \times [\eta_0,\infty)$ and $\lim_{\varepsilon\to 0} T_\varepsilon^* = T^* = \infty$, where Ω_2 is an arbitrary compact subset of Ω. Then there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$,

$$\big\|u_{\varepsilon xx}(x,\eta_0)\big\|_{L^\infty(\Omega_2)} \leqslant \big\|u_{xx}(x,\eta_0)\big\|_{L^\infty(\Omega_2)} + 1 \leqslant \big\|u_{xx}(x,\eta_0)\big\|_{L^\infty(\Omega)} + 1 = C_0. \tag{4.9}$$

It is obviously that $C_0 = C_0(\eta_0)$ depends only on η_0. Thus from (4.8) and (4.9), it follows that $u_{xx} \leqslant C_0$ in any compact subset Ω_2 of Ω for all $t \geqslant \eta_0$. □

Lemma 4.3. *Let the assumptions in Lemma* 4.2 *hold, let $\eta_0 < T_1^*/2$ be any given small positive constant, and let v be the function defined by*

$$v(x,t) = u(x,t) - \frac{C_0}{2}(x-a_1)(x-b), \tag{4.10}$$

where $C_0 = C_0(\eta_0)$ is given by Lemma 4.2*. Then there exists a positive constant C_1 such that*

$$\forall t > \eta_0, \quad v\big(x_0(t),t\big) \geqslant C_1\big\|v(\cdot,t)\big\|_{L^\infty(\Omega)}. \tag{4.11}$$

Proof. Let K be the compact subset of Ω given by hypothesis (H_1) such that for all $t \geqslant 0$, $x_0(t) \in K$, and let $\alpha = \operatorname{dist}(K, \partial\Omega) > 0$, then the definition of the function v and the conclusion of Lemma 4.2 imply that

$$v_{xx} = u_{xx} - C_0 \leqslant 0 \tag{4.12}$$

holds for all $t \geqslant \eta_0$ on any compact subset of Ω. Hence we know that v as a function of x is concave in Ω. Now, let us denote by $X(t) \in \Omega$ a point such that $\|v(\cdot,t)\|_{L^\infty(\Omega)} = v(X(t),t)$. Suppose first that $a_1 < x_0(t) \leqslant X(t)$. Using the concavity property of v in x, we obtain for all $t \geqslant \eta_0$,

$$\frac{v(x_0(t),t) - v(a_1,t)}{x_0(t) - a_1} \geqslant \frac{v(X(t),t) - v(a_1,t)}{X(t) - a_1}.$$

In virtue of $v(a_1,t) = 0$ by (4.10), from the above inequality we get for all $t \geqslant \eta_0$,

$$v\big(x_0(t),t\big) \geqslant \frac{x_0(t) - a_1}{X(t) - a_1} v\big(X(t),t\big) \geqslant \frac{\alpha}{b - a_1} v\big(X(t),t\big). \tag{4.13}$$

On the other hand, if $X(t) \leqslant x_0(t) < b$, we can also use the concavity property of v in x, and then we obtain for all $t \geqslant \eta_0$,

$$\frac{v(b,t) - v(X(t),t)}{b - X(t)} \geqslant \frac{v(b,t) - v(x_0(t),t)}{b - x_0(t)}.$$

In virtue of $v(b,t) = 0$ by (4.10), from the above inequality we again get for all $t \geqslant \eta_0$,

$$v\big(x_0(t),t\big) \geqslant \frac{b - x_0(t)}{b - X(t)} v\big(X(t),t\big) \geqslant \frac{\alpha}{b - a_1} v\big(X(t),t\big). \tag{4.14}$$

Choosing $C_1 = \alpha/(b - a_1)$, then from inequalities (4.13) and (4.14), we obtain (4.11). □

Proof of Theorem 1.2. From Lemma 4.3 and the positivity of $u(x,t)$, we know that for all $t \geqslant \eta_0$,

$$\begin{aligned} u\big(x_0(t),t\big) &= v\big(x_0(t),t\big) + \frac{C_0}{2}\big(x_0(t) - a_1\big)\big(x_0(t) - b\big) \\ &\geqslant \left(C_1 \big\|v(\cdot,t)\big\|_{L^\infty(\Omega)} - \frac{C_0}{2}(b - a_1)^2\right)_+ \\ &\geqslant \left(C_1 \big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} - \frac{C_0}{2}(b - a_1)^2\right)_+, \end{aligned} \tag{4.15}$$

where C_0, η_0 and $C_1 > 0$ are three constants given in Lemmas 4.2 and 4.3, respectively. Since f is increasing and convex, we have

$$\begin{aligned} f\left(\frac{C_1}{2}\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}\right) &\leqslant f\left(\frac{C_1}{2}\left(\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} - \frac{C_0}{2C_1}(b - a_1)^2\right)_+ + \frac{C_0}{4}(b - a_1)^2\right) \\ &\leqslant \frac{1}{2} f\left(\left(C_1\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} - \frac{C_0}{2}(b - a_1)^2\right)_+\right) \\ &\quad + \frac{1}{2} f\left(\frac{C_0}{2}(b - a_1)^2\right). \end{aligned}$$

This inequality combining with (4.15) yields for all $t \geqslant \eta_0$,

$$\begin{aligned}
f\big(u\big(x_0(t),t\big)\big) &\geqslant f\left(\left(C_1\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}-\frac{C_0}{2}(b-a_1)^2\right)_+\right)\\
&\geqslant 2f\left(\frac{C_1}{2}\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}\right)-f\left(\frac{C_0}{2}(b-a_1)^2\right)\\
&\geqslant 2f\left(\frac{C_1}{2}u(x,t)\right)-C_2,
\end{aligned}$$

where $C_2=f(\frac{C_0}{2}(b-a_1)^2)$. Using the equation in (1.1) and the above inequality, we obtain for all $t\geqslant\eta_0$,

$$u_t=u^r\big(\Delta u+af\big(u\big(x_0(t),t\big)\big)\big)\geqslant u^r\left(\Delta u+2af\left(\frac{C_1}{2}u(x,t)\right)-aC_2\right). \tag{4.16}$$

Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and φ_1 be the corresponding eigenfunction such that $\varphi_1(x)>0$ in Ω and $\int_\Omega\varphi_1(x)\,dx=1$. Set

$$U(t)=\int_\Omega\frac{1}{1-r}u^{1-r}(x,t)\varphi_1(x)\,dx,\quad t\geqslant 0, \tag{4.17}$$

then by using (4.16), the convexity property of f and the Jensen's inequality, we get for all $t\geqslant\eta_0$,

$$\begin{aligned}
U'(t)&=\int_\Omega u^{-r}u_t\varphi_1\,dx\\
&\geqslant\int_\Omega\Delta u\varphi_1\,dx+2a\int_\Omega f\left(\frac{C_1}{2}u(x,t)\right)\varphi_1(x)\,dx-aC_2\\
&\geqslant-\lambda_1\int_\Omega u(x,t)\varphi_1(x)\,dx+2af\left(\frac{C_1}{2}\int_\Omega u(x,t)\varphi_1(x)\,dx\right)-aC_2.
\end{aligned} \tag{4.18}$$

Let $w=u^{1-r}$, then $U(t)=\frac{1}{1-r}\int_\Omega w(x,t)\varphi_1(x)\,dx$. Since $\frac{1}{1-r}>1$, using the Jensen's inequality again, we get

$$\int_\Omega u(x,t)\varphi_1(x)\,dx=\int_\Omega w^{\frac{1}{1-r}}(x,t)\varphi_1(x)\,dx\geqslant\left(\int_\Omega w(x,t)\varphi_1(x)\,dx\right)^{\frac{1}{1-r}}. \tag{4.19}$$

Since $f(s)$ satisfies (H_3) and (H_4), from the proof of Theorem 3.2 we know that $\lim_{s\to\infty}f(s)=\infty$ and $\lim_{s\to\infty}\frac{f(s)}{s}=\infty$. And then there exists a positive constant $s_4\geqslant s_0$ such that $\frac{f(s)}{s}>\frac{2\lambda_1}{aC_1}$ for all $s\geqslant s_4$. Set $C_3=\max\{\frac{2}{C_1}s_4,\frac{2}{C_1}f^{-1}(C_2)\}$ and $C_4=\max\{1,C_3^{1-r}\}$, where f^{-1} is the inverse function of f in $(0,\infty)$. If there exists a $T'\geqslant\eta_0$ such that

$$\int_\Omega w(x,T')\varphi_1(x)\,dx>C_4, \tag{4.20}$$

then by using (4.19), we get

$$\int_\Omega u(x,T')\varphi_1(x)\,dx\geqslant\left(\int_\Omega w(x,T')\varphi_1(x)\,dx\right)^{\frac{1}{1-r}}>C_3.$$

Then from the above inequality and the choice of C_3 we know that there exists an $\varepsilon_0 > 0$ such that

$$-\lambda_1 \int_\Omega u(x,T')\varphi_1(x)\,dx + 2af\left(\frac{C_1}{2}\int_\Omega u(x,T')\varphi_1(x)\,dx\right) - aC_2$$
$$\geqslant \varepsilon_0 f\left(\frac{C_1}{2}\int_\Omega u(x,T')\varphi_1(x)\,dx\right) > 0. \tag{4.21}$$

Utilizing (4.18), (4.21) and (4.19), for all $t \geqslant T'$, we obtain

$$U'(t) \geqslant \varepsilon_0 f\left(\frac{C_1}{2}\left(\int_\Omega w(x,t)\varphi_1(x)\,dx\right)^{\frac{1}{1-r}}\right) > 0.$$

Noticing the choice of C_4 and using (4.20) and the nondecreasing property of f, we can easily get from the above inequality that for all $t \geqslant T'$,

$$U'(t) \geqslant \varepsilon_0 f\left(\frac{C_1}{2}\int_\Omega w(x,t)\varphi_1(x)\,dx\right) = \varepsilon_0 f\left(\frac{C_1}{2}(1-r)U(t)\right). \tag{4.22}$$

Integrating the inequality (4.22) from T' to T^*, we obtain

$$\int_{(1-r)C_1U(T')/2}^{(1-r)C_1U(T^*)/2} \frac{ds}{f(s)} \geqslant \frac{C_1}{2}(1-r)\varepsilon_0\big(T^* - T'\big),$$

that is,

$$T^* \leqslant T' + \frac{2}{C_1(1-r)\varepsilon_0}\int_{\frac{C_1}{2}(1-r)U(T')}^{\infty} \frac{ds}{f(s)} < \infty,$$

which means that $u(x,t)$ blows up in finite time under condition (4.20).

On the other hand, it follows from the definition of α given in the proof of Lemma 4.3 that $0 < \alpha \leqslant \frac{b-a_1}{2}$. Using the concavity property of $v(x,t)$ obtained also in the proof of Lemma 4.3, we get, for any $x \in [a_1 + \frac{\alpha}{2}, a_1 + \alpha]$, if $a_1 < x \leqslant x_0(t)$ then for all $t \geqslant \eta_0$,

$$u(x,t) + \frac{C_0}{8}(b-a_1)^2 \geqslant v(x,t) \geqslant \frac{x-a_1}{x_0(t)-a_1}v\big(x_0(t),t\big) \geqslant \frac{\alpha}{2(b-a_1)}u\big(x_0(t),t\big),$$

and if $x_0(t) \leqslant x < b$ then for all $t \geqslant \eta_0$,

$$\begin{aligned} u(x,t) + \frac{C_0}{8}(b-a_1)^2 \geqslant v(x,t) &\geqslant \frac{x-b}{x_0(t)-b}v\big(x_0(t),t\big) \\ &\geqslant \frac{a_1+\alpha-b}{a_1-b}v\big(x_0(t),t\big) \geqslant \frac{\alpha}{b-a_1}v\big(x_0(t),t\big) \\ &\geqslant \frac{\alpha}{2(b-a_1)}u\big(x_0(t),t\big). \end{aligned}$$

Hence we have

$$u(x,t) \geqslant \frac{\alpha}{2(b-a_1)} u\big(x_0(t),t\big) - \frac{C_0}{8}(b-a_1)^2, \quad x \in \left[a_1+\frac{\alpha}{2}, a_1+\alpha\right],\ t \in \big[\eta_0, T^*\big). \tag{4.23}$$

We denote $k_0 = \inf_{x\in[a_1+\frac{\alpha}{2},a_1+\alpha]} \varphi_1(x)$, choose

$$B > \frac{2(b-a_1)}{\alpha}\left[\left(\frac{2C_4}{k_0\alpha}\right)^{\frac{1}{1-r}} + \frac{C_0}{8}(b-a_1)^2\right]$$

and set $A = af(B)$. We assume by contradiction that

$$\limsup_{t\to\infty} \big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} = \infty,$$

from Lemma 4.1 we know that there exists a positive constant $t_0 \geqslant \eta_0$ such that

$$af\big(u\big(x_0(t_0),t_0\big)\big) \geqslant A,$$

then using the nondecreasing property of f, we obtain $u(x_0(t_0),t_0) \geqslant B$. And then from (4.23) and the choice of B, we obtain

$$\begin{aligned}
\int_\Omega u^{1-r}(x,t_0)\varphi_1(x)\,dx &\geqslant k_0 \int_{a_1+\frac{\alpha}{2}}^{a_1+\alpha} u^{1-r}(x,t_0)\,dx \\
&\geqslant k_0 \int_{a_1+\frac{\alpha}{2}}^{a_1+\alpha} \left[\frac{\alpha}{2(b-a_1)} u\big(x_0(t_0),t_0\big) - \frac{C_0}{8}(b-a_1)^2\right]^{1-r} dx \\
&\geqslant \frac{\alpha}{2} k_0 \left[\frac{\alpha}{2(b-a_1)} B - \frac{C_0}{8}(b-a_1)^2\right]^{1-r} > C_4,
\end{aligned}$$

which implies, by taking $T' = t_0$ in the above discussion, that $T^* < \infty$, a contradiction. And therefore $\sup_{t\geqslant 0} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. $\square$

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